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Magneto-hydrostatic fields

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With 1 figure in the text

I. Introduction

In an electrically conducting liquid which is immersed in a magnetic field the interaction between electromagnetic and hydrodynamic forces may give rise to a wave-motion.[1] The characteristic feature of these magneto-hydrodynamic waves is the equipartition of kinetic and magnetic energy. The "guiding" magnetic field is in this case supposed to be of external origin. If the total magnetic field is caused by currents in the liquid, the phenomena become much more complicated and have scarcely been studied before. In general an electric current starts a motion in a liquid. The purpose of this paper is to show that in special cases magnetic fields may exist in a liquid at rest. Such "*magneto-hydrostatic*" fields may be important in the theory of the sun's and the earth's magnetism.[2] Possibly the cosmic ray trapping fields may be of this type.[3]

In the following some properties of these fields are discussed. The stability of these fields, which is important for the applications, is not treated in this paper.

II. General properties

A hydrostatic equilibrium in a liquid is possible only if the force acting upon it, may be derived from a potential Φ. In a medium with the permeability $\mu = 1$, a current density $\bar{i}$ and a magnetic field $\bar{H}$ give a volume force = $= \frac{1}{c}\,\bar{i} \times \bar{H} = \frac{1}{4\pi}\,\mathrm{curl}\,\bar{H} \times \bar{H}$.

The magneto-hydrostatic fields are accordingly described by

$$\mathrm{curl}\,\bar{H} \times \bar{H} = \nabla \Phi \tag{1}$$

if the conductivity is infinite (no damping).

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[1] H. Alfvén, Ark. f. Mat. Astr. o. Fysik Bd *29 B*, No. 2 (1942). Further ref. in Cosmical Electrodynamics (Oxford, 1950) by the same author.

[2] H. Alfvén, Tellus (In press).

[3] R. D. Richtmyer and E. Teller, Phys. Rev. *75*, 1729 (1949). H. Alfvén, ibid. *75*, 1732 (1949), *77*, 375 (1950).

We may distinguish between forcefree fields with $\nabla \Phi = 0$ and pressure-balanced fields with $\nabla \Phi \neq 0$ and $\nabla \Phi = 4\pi \nabla p$, where p means the hydrostatic pressure.

III. Forcefree fields

The forcefree fields are characterized by

$$\operatorname{curl} \bar{H} \times \bar{H} = 0 \tag{2}$$

which means that in regions, where neither the magnetic field nor the current vanishes, the current must flow parallel to the magnetic field. Eq. (2) may then be replaced by the linear relation

$$\operatorname{curl} \bar{H} = \alpha(\bar{r}) \cdot \bar{H} \tag{3}$$

where $\alpha(\bar{r})$ may be any space-function satisfying

$$\nabla \alpha \cdot \bar{H} = 0. \tag{4}$$

The last equation follows from (3) on observing that

$$\operatorname{div} \bar{H} = 0. \tag{5}$$

It may be interesting to investigate if a forcefree current distribution within a closed volume may give rise to a magnetic field outside this volume. In the outer region we assume

$$\operatorname{curl} \bar{H} = 0. \tag{6}$$

As (5) is valid in the whole space the solution of (6) may be written

$$\bar{H} = \nabla \psi \tag{7}$$

$$\nabla^2 \psi = 0. \tag{8}$$

From (6) we infer that the normal component of the current

$$(\operatorname{curl} \bar{H})_n = 0 \tag{9}$$

at the boundary surface. If $\alpha \neq 0$ at the boundary eq. (3) and (9) tell us that

$$H_n = 0. \tag{10}$$

If $\alpha = 0$ at the surface, $\nabla \alpha$ is perpendicular to the surface and then we see from eq. (4) that eq. (10) is valid in this case too. As $\bar{H}$ must be continuous through the surface (no surface current) we have $\partial \psi / \partial n = 0$ at the boundary. The total outer field is supposed to be due to the forcefree currents and must vanish at least like a dipole field at infinity, which means that $\partial \psi / \partial n = 0$ at the whole boundary of the outer region. According to a well known theorem on harmonic functions, we then have $\nabla \psi = 0$ in the whole

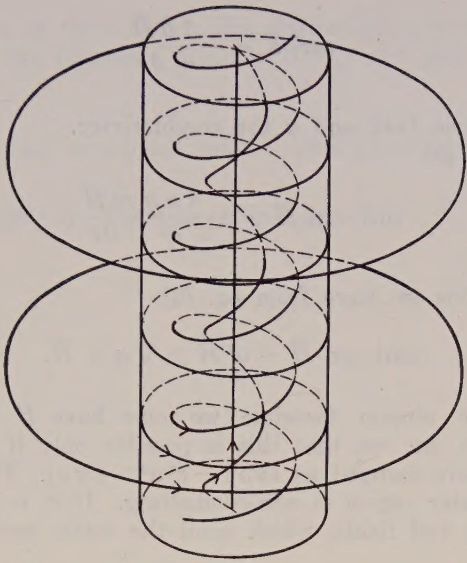

Fig. 1. Force-free field in an infinite cylinder.

outer region if ψ is a one-valued function. If the region is not simply connected this theorem is no more valid and an outer field is possible. The conclusion is that if an outer dipole field is due to a forcefree field in a sphere, the current is contained in a toroidal shell within the sphere.

As an example we study the field in a circular cylinder of infinite length. Introducing cylinder-coordinates (r, φ, z) and putting $\bar{H} = /0, H_\varphi(r), H_z(r)/$ we get from eq. (3) under the simplifying assumption $\alpha = \text{const.}$

$$\alpha H_\varphi = -\frac{\partial H_z}{\partial r} \tag{11}$$

$$\alpha H_z = \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi). \tag{12}$$

The solution is

$$H_z = A \cdot J_0(\alpha r) \tag{13}$$

$$H = A \cdot J_1(\alpha r). \tag{14}$$

If $H_z = 0$ at the surface $(r = R)$ of the cylinder the outer field is a tangential field, decreasing as r^{-1}. The lines of force (and current) are shown in fig. 1.

Physically more important is the toroidal field. The general solution involves mathematical difficulties. In a very thin torus the field is essentially the same as in the long cylinder. The outer field is at great distance a dipole field.

If the conductivity, σ, is finite the time variation of these fields is described by

$$\operatorname{curl} \bar{H} = \frac{4\pi}{c}\bar{i} = \frac{4\pi\sigma}{c}\cdot\bar{E} \tag{15}$$

$$\operatorname{curl} \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \tag{16}$$

where $\bar{E}$ is the electric field and σ the conductivity.

Eliminating $\bar{E}$ we get

$$\operatorname{curl}\operatorname{curl} \bar{H} = -\frac{4\pi\sigma}{c^2}\cdot\frac{\partial \bar{H}}{\partial t}. \tag{17}$$

For the forcefree fields we have from eq. (3).

$$\operatorname{curl}\operatorname{curl} \bar{H} = \alpha^2 \bar{H} + \nabla \alpha \times \bar{H}. \tag{18}$$

If the field is to remain forcefree we must have $H = H_0(\bar{r}) \exp(-\lambda t)$. From (17) and (18) we see that this is possible only if α is a constant. In this case the fields are damped as $\exp(-c^2 \alpha^2 t / 4\pi\sigma)$. This criterion is significant only if the outer region is non-conducting. If it is conducting, we may get induced currents and fields, which spoil the static properties.

IV. Pressure-balanced fields

From eq. (1) we see that the pressure-balanced fields are characterized by the fact that the lines of force and current are situated in the surfaces $\Phi = \text{const}$. Using the identity

$$\operatorname{curl} \bar{H} \times \bar{H} = (\bar{H}\cdot\nabla)\bar{H} - \nabla\frac{H^2}{2} \tag{19}$$

(1) may be written

$$(\bar{H}\cdot\nabla)\bar{H} = \nabla\left(\Phi + \frac{H^2}{2}\right). \tag{20}$$

Denoting the arc-length along a line of force by s and the unit tangent, normal and binormal to this curve by $\hat{t}$, $\hat{n}$, $\hat{b}$ we may write eq. (20)

$$H\frac{\partial}{\partial s}(H\hat{t}) = \hat{t}\frac{\partial}{\partial s}\left(\frac{H^2}{2}\right) + \hat{n}\cdot\frac{H^2}{R} = \nabla\left(\Phi + \frac{H^2}{2}\right) \tag{21}$$

where R means the radius of principal curvature.

Resolving ∇ along $\hat{t}$, $\hat{n}$, $\hat{b}$ we get

$$\frac{H^2}{R} = \frac{\partial}{\partial n}\left(\Phi + \frac{H^2}{2}\right) \tag{22}$$

$$0 = \frac{\partial}{\partial b}\left(\Phi + \frac{H^2}{2}\right). \tag{23}$$

The magnetic lines of force behave like elastic strings with a tension $= H^2/4\pi$ and acted upon by the magnetic pressure $H^2/8\pi$ and the hydrostatic pressure $p = \Phi/4\pi$.

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